Approximating Travelling Waves by Equilibria of Non Local Equations

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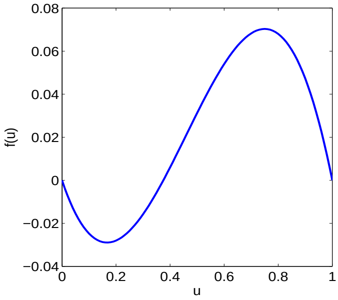
Introduction

Consider the initial value problem for the unknown $u = u(x,t)$,

$$\begin{cases} u_t = u_{xx} + f(u), & -\infty < x < +\infty, \quad t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

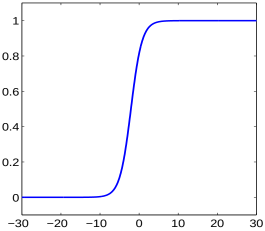
with $f \in C^1(\mathbb{R})$ with $f(0) = f(1) = f(\alpha) = 0$, $f'(0) < 0$, $f'(1) < 0$ (*bistable case*), for some $0 < \alpha < 1$, u_0 piecewise continuous and $0 \leq u_0 \leq 1$.



Travelling Waves

Special solutions of the form $u(x,t) = \Phi(x - ct)$ for some $c \in \mathbb{R}$, with finite different limits at $\pm\infty$, so that

$$\begin{cases} \Phi''(\xi) + c\Phi'(\xi) + f(\Phi(\xi)) = 0, & -\infty < \xi < +\infty, \\ 0 \le \Phi \le 1, \quad \Phi(-\infty) = 0,\ \Phi(+\infty) = 1, & \Phi' > 0. \end{cases}$$



Also with $\Phi(-\infty) = 0,\ \Phi(+\infty) = 1$.

If $\Phi(x - ct)$ is a travelling wave for the original equation, so is $\Phi(x - ct - a)$, $a \in \mathbb{R}$ (invariance of the equation under translation).

The approach of solutions to travelling waves

Theorem (Fife & McLeod 1977) There exists a unique $c \in \mathbb{R}$ and a unique (except for translation) monotone solution Φ of

$$\Phi'' + c\Phi' + f(\Phi) = 0,$$

with $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$, so that $(x,t) \to \Phi(x - ct - a)$ is a solution of the parabolic equation $\forall a \in \mathbb{R}$.
Suppose that u_0 is piecewise continuous, $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$, and

$$\limsup_{x \to -\infty} u_0(x) < \alpha, \qquad \liminf_{x \to \infty} u_0(x) > \alpha.$$

Then, there exists $x_0 \in \mathbb{R}$, $K, \omega > 0$, such that

$$|u(x,t) - \Phi(x - ct - x_0)| < Ke^{-\omega t}, \qquad x \in \mathbb{R}, \quad t > 0.$$

Difficulties for the Numerical Approximation

- Need of truncating the computational spatial domain. On the one hand, this affects the asymptotic behavior of the solution. On the other, the solution will eventually leave the selected domain while travelling left or right.
- Natural approach: The change of variables $u(x,t) = v(x-ct,t)$.
- BUT in general c is not known a priori, but depends on u via the nonlinear term f.

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Idea [Beyn & Thummler, SIADS 2004]

A suitable change of variables $u(x,t) = v(x - \gamma(t), t)$, with $\gamma(t)$ an additional unknown. Then

$$v_t = v_{xx} + \gamma' v_x + f(v),$$

and it is needed one extra equation to compensate for the additional degree of freedom: "Phase condition" to compute at the same time v and γ (or γ').

Change of Variables and Phase Condition

The change of variables $u(x,t) = v(x-\gamma(t),t)$ leads to the equation

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + \gamma'(t)v_x(x,t) + f(v(x,t)), & -\infty < x < +\infty, \ t > 0, \\ v(x,0) = u_0(x-\gamma(0)). \end{cases}$$

Phase condition: How to choose a link between v and γ?

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Multiplication in

$$\Phi''(\xi) + c\Phi'(\xi) + f(\Phi(\xi)) = 0, \quad -\infty < \xi < \infty,$$

by Φ' and integration along the real line yields

$$c = -\frac{\langle f(\Phi), \Phi' \rangle}{\langle \Phi', \Phi' \rangle} = -\frac{F(1)}{\|\Phi'\|^2_{L^2(\mathbb{R})}},$$

with $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{R})$.

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Take

$$\gamma_v(t) = -\int_0^t \frac{F(1)}{\|v_x(\cdot, s)\|^2_{L^2(\mathbb{R})}} \, ds.$$

In particular $v(x,0) = u_0(x)$.

The modified equation

Elimination of γ' leads to the nonlocal equation

$$\begin{cases} v_t = v_{xx} - \dfrac{F(1)}{\|v_x\|^2_{L^2(\mathbb{R})}}\, v_x + f(v), \quad -\infty < x < +\infty, \quad t > 0, \\ v(0) = u_0. \end{cases}$$

If $|v_t| \to 0$ as $t \to \infty$, *we expect* that

$$\lambda_v(t) := \gamma'_v(t) = -\frac{F(1)}{\|v_x(\cdot, t)\|^2_{L^2(\mathbb{R})}} \to \lambda^*$$

and *hopefully* $\lambda^* = c$.

Example: The Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-\alpha), \qquad u(x,0) = u_0(x), \quad \alpha \in (0,1).$$

Explicit traveling wave solution

$$u(x,t) = \bar{v}(x - ct)$$

with

$$\bar{v}(x) = \frac{1}{1 + \exp(-\frac{x}{\sqrt{2}})}, \qquad c = -\sqrt{2}\left(\frac{1}{2} - \alpha\right).$$

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Choose $\alpha = 1/4$, so that $c \approx -0.3536$, fix a finite spatial computational domain $[-J, J]$, $J > 0$ and set reasonable boundary conditions. Either of Dirichlet type:

$$u(-J,t) = 0, \qquad u(J,t) = 1, \qquad t > 0,$$

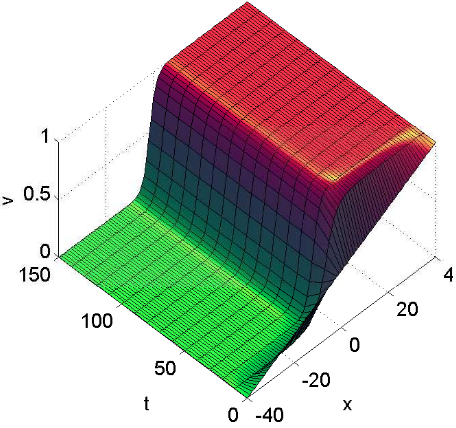
or of Neumann type:

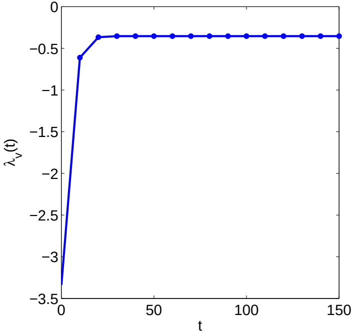
$$u_x(-J,t) = 0, \qquad u_x(J,t) = 0, \qquad t > 0.$$

Example 1

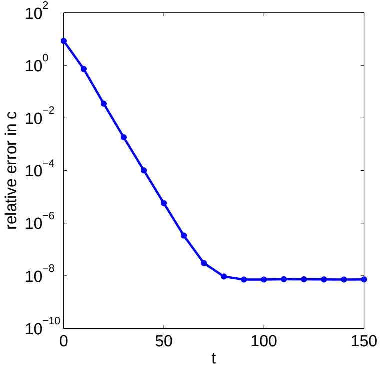
Take a linear initial data satisfying the Dirichlet type boundary conditions

$$u_0(x) = \frac{1}{2J}(x+J), \qquad -J \leq x \leq J.$$





Example 1. Error in the approximation of c



Evolution of the relative error in the approximation of c, i.e., $|\lambda_v(t) - c|/c$.
Left: $J = 40$, $\Delta x = 0.1$. *Right:* $J = 40$, $\Delta x = 0.025$.

Convergence to a stationary state of the nonlocal problem

Theorem (J. Arrieta, MLF, E. Zuazua, 2010) *Under the hypotheses of* f *and* u_0 *as above (Fife & McLeod) and assuming further that* $\partial_x u_0 \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, *the nonlocal problem is well-posed. Its solution* v *is given by*

$$v(x,t) := u(x + \gamma_u(t), t),$$

where $\gamma_u(t) := -\int_0^t \frac{F(1)}{\|u_x(\cdot, s)\|^2_{L^2(\mathbb{R})}}\, ds$, *and* u *is the solution to the original Cauchy problem.*

There also exist $x_1 \in \mathbb{R}$ *and positive constants* C_1, C_2 *and* ω *such that*

$$|\lambda_v(t) - c| \leq C_1 e^{-\omega t}, \qquad t > 0,$$

for $\lambda_v = \gamma_v'$, *and*

$$|v(x,t) - \Phi(x - x_1)| < C_2 e^{-\omega t}, \qquad x \in \mathbb{R}, \quad t > 0.$$

The nonlocal problem in a bounded domain

For practical use the computational domain will be always truncated to a bounded interval (a, b) and we need to supplement boundary conditions.

We study the nonlocal problem in (a, b) with "consistent" boundary conditions of Dirichlet type:

$$\begin{cases} v_t = v_{xx} + \lambda(v)\, v_x + f(v), \quad a < x < b, \quad t > 0, \\ v(a, t) = 0, \quad v(b, t) = 1, \\ v(0) = v_0, \end{cases}$$

where now

$$\lambda(v) = -\frac{F(1)}{\|v_x\|^2_{L^2(a,b)}}$$

and we assume $0 \le v_0 \le 1$.

Well posedness and stationary problem

Proposition (J. Arrieta, MLF, E. Zuazua, 2010) The nonlocal IBVP is locally well posed: for any initial data $u_0 \in H^1(a, b)$ with $u_0(a) = 0$, $u_0(b) = 1$ there exists a $T = T(u_0)$ and a unique classical solution $u(x, t)$ defined for the time interval $[0, T(u_0))$.
Moreover, if $0 \leq u_0 \leq 1$, then the solution $u(x, t)$ is globally defined and it also satisfies $0 \leq u(x, t) \leq 1$.

Remark: A general comparison result for the nonlocal equation is not available, only comparison with the constants.

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Associated stationary problem:

$$\begin{cases} \Phi'' + \lambda^* \Phi' + f(\Phi) = 0, & a < x < b, \\ \Phi(a) = 0, \quad \Phi(b) = 1, \end{cases}$$

for $\lambda^* = \lambda(\Phi) = -\dfrac{F(1)}{\|\Phi_x\|^2_{L^2(a,b)}}$.

Stationary solutions

Characterization: A solution of the local boundary problem

$$\begin{cases} \Phi'' + \lambda^* \Phi' + f(\Phi) = 0, & a < x < b, \\ \Phi(a) = 0, \quad \Phi(b) = 1 \end{cases}$$

is a solution of the nonlocal boundary problem

$$\begin{cases} \Phi'' - \dfrac{F(1)}{\|\Phi'\|^2_{L^2(a,b)}} \Phi' + f(\Phi) = 0, & a < x < b, \\ \Phi(a) = 0, \quad \Phi(b) = 1, \end{cases}$$

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Proposition (Phase plane techniques, shooting method)

The stationary nonlocal problem has one and only one equilibrium Φ_r for $r = b - a$. Moreover $\Phi_r'(\cdot) > 0$.

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- $a(r) \to -\infty$ and $b(r) \to +\infty$.
- If we extend the value of Φ_r by 0 to the left of $a(r)$ and by 1 to the right of $b(r)$ and we normalize the travelling wave so that also $\Phi_\infty(0) = 1/2$, then

$$\|\Phi_r - \Phi_\infty\|_{W^{1,\infty}(R)} + \|\Phi_r' - \Phi_\infty'\|_{L^2(R)} \to 0.$$

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- STABILITY: the solution of the nonlocal IBVP in (a, b) approaches the stationary solution as $t \to \infty$ (and thus the TW for r large enough). Moreover, the convergence is exponentially fast in time.

Stability of the stationary solution

Spectrum of the linearization around the equilibrium Φ_r:

$$L^r w = w'' + \lambda(\Phi_r)w' + f'(\Phi_r)w + \pi_r(w)\Phi_r',$$

where

$$\pi_r(w) = -\frac{2\lambda(\Phi_r)}{\|\Phi_r'\|^2_{L^2(a,b)}} \cdot \int_a^b \Phi_r'(x)w'(x)dx,$$

in (a, b) with homogeneous Dirichlet boundary conditions.

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$$L^r w = L_0^r w + \pi_r(w)\Phi_r',$$

where L_0^r is the second order elliptic operator

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with $\sigma(L_0^r) \subset \{z \in \mathbb{C},\ \mathrm{Re}\, z < 0\}$.

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We show that $\sigma(L^r) \subset \{z \in \mathbb{C},\ \mathrm{Re}\, z < 0\}$, for large enough $r > 0$.

References

There are several results in the literature that analyze the spectra of operators of the form

$$w \to w'' + b(x)w' + c(x)w + d(x)\int_a^b e(x)w(x)dx.$$

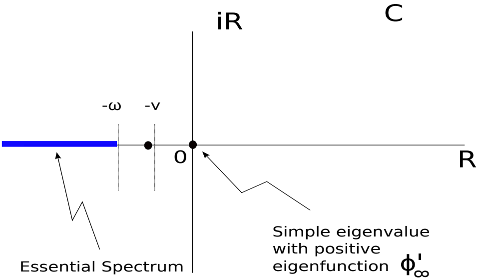
Fiedler & Polacik (1990); Freitas (94), (99); Davidson & Dodds (2006)

All of them conclude that the spectra maybe very different from the spectra of the local operator and may contain complex eigenvalues.

None of these results can be applied to our operators to conclude that the spectra is "stable".

Steps of the proof

- In the whole line: $\sigma(L^\infty) = \sigma(L_0^\infty)$.
  KEY: Φ'_∞ is the eigenfunction of the local operator L_0^∞ associated to the eigenvalue 0 and $\pi_\infty(\Phi'_\infty) = 0$.



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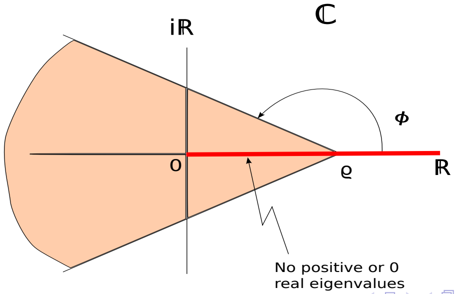
- In a bounded interval:
  - $\sigma(L^r)$ is made up only of eigenvalues and it is contained in a uniform sector $\Sigma_{\rho,\phi} = \{z \in \mathbb{C} : |Arg(z-\rho)| > \phi\}$ with $\rho \in \mathbb{R}$ and $\phi \in (\pi/2, \pi)$
  - Any real eigenvalue of $\sigma(L^r)$ is strictly negative.
  - We need to exclude the possibility of having complex eigenvalues with positive real part.



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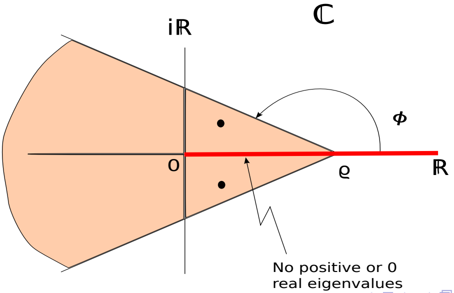

Perturbation argument: Discrete Convergence of Operators

Theory developed by Vainnikko (70's). It allows to show that

$$\sigma(L^r) \cap \{z \in \mathbb{C},\ \mathrm{Re}\, z > -\nu\} \to \sigma(L^\infty) \cap \{z \in \mathbb{C},\ \mathrm{Re}\, z > -\nu\}$$

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a simple eigenvalue. Then for r large enough

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is a simple eigenvalue and $z(r) \to 0$ as $r \to +\infty$.
Necessarily $z(r) \in \mathbb{R}$ and then $z(r) < 0$.

To obtain the discrete convergence of the operators above we use:

- The problem is 1-dimensional
- Away from the essential spectrum we have "exponential dichotomies" for the appropriate linear operators and the operators are "Fredholm of index 0".
- The convergence of Φ_r to Φ_∞

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OPEN: This approach has been proposed (W.J. Beyn et al., 2004-2008) in the more general framework of evolution equations which are equivariant under the action of some Lie group.

- Validation of the approach for other types of nonlinearities and also for systems.
- Validation of the approach for the computation of more general relative equilibria of evolution equations.